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POSITIVE (SEMI-) DEFINITE MATRICES
AND MATHEMATICAL PROGRAMMING

by

George B. Dantzig and Richard W. Cottle

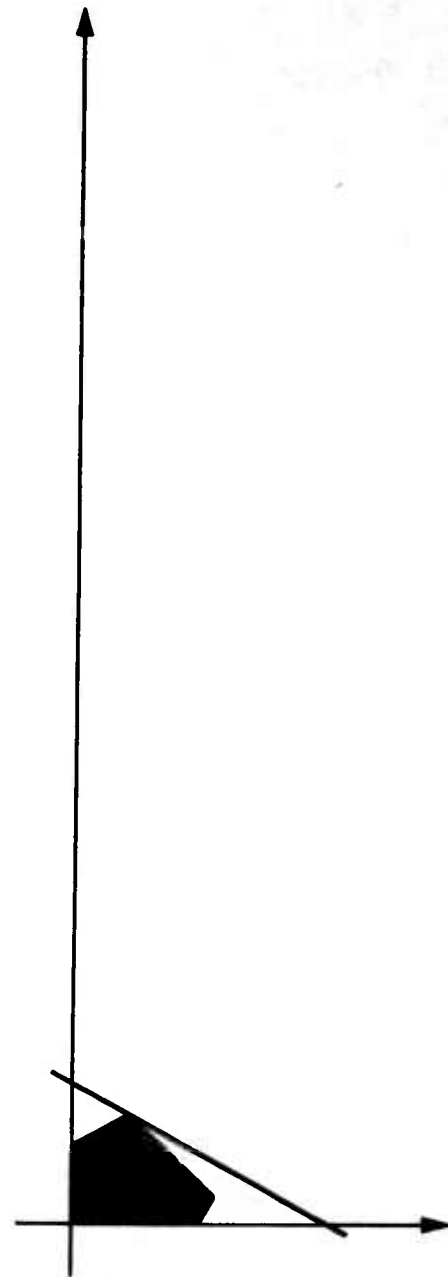
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POSITIVE (SEMI-) DEFINITE MATRICES AND MATHEMATICAL PROGRAMMING

BY

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ABSTRACT

This paper first considers a general mapping w of points z in R^N into points $w(z)$ in the same space. It is shown that classical mathematical programming problems can all be restated as finding among those vectors $z \geq 0$ which map into $w \geq 0$, one which minimizes $z^T w$. For a differentiable mapping w with a positive semi-definite Jacobian matrix the obviously sufficient ("complementary slackness") condition $z^T w = 0$ for a minimum is shown to be necessary.

Next considered is the case which includes quadratic and linear programming: $w = Mz + q$ where q is a fixed vector and M is positive (semi-) definite with constant coefficients. It is noted that the definiteness property is preserved for any equivalent system generated by an interchange of a subset of corresponding complements of z and w or what is the same thing by a "block" pivot on a principal minor of M . Since this result is related to recent ones obtained by A. Tucker and P. Wolfe, it is planned to incorporate its proof in a separate joint paper.

Finally, a simple constructive solution of quadratic and linear programs is presented. In a sense the simplex method for linear programs and the results of Barankin, Dorfman, Wolfe, Beale, and Markowitz on quadratic programs re-appear here but are curiously free of primal-dual structure. Complementary pairs of variables play a key role.

POSITIVE (SEMI-) DEFINITE MATRICES AND MATHEMATICAL PROGRAMMING

I. The Optimality Criterion and Its Application

Let $w(z) = (w_1(z), \dots, w_N(z))^T$ be an arbitrary vector function mapping R^N into itself. Given the mathematical program

$$(1) \quad \begin{aligned} & \text{Minimize} && z^T w(z) \\ & \text{subject to} && w(z) \geq 0 \\ & && z \geq 0 \end{aligned}$$

an obviously sufficient condition for the optimality of a feasible vector z^0 is

$$(2) \quad z^{0T} w(z^0) = 0$$

which implies that for each $j = 1, \dots, N$, at most one of z_j^0 and $w_j(z^0)$ may be positive. Under certain circumstances, (2) is also a necessary condition of the optimality of z^0 in (1). For instance, assume that $w(z)$ is a differentiable mapping. Denote by $[\partial w / \partial z]$ the transposed Jacobian matrix, $[\partial w_j / \partial z_i]$. (For this notation and terminology, see [1].) It will furthermore be assumed that the Kuhn-Tucker constraint qualification [8] is satisfied by the set

$$\{z \mid w(z) \geq 0, z \geq 0\}.$$

Let $[\partial w / \partial z]^0$ denote $[\partial w / \partial z]$ with each of its entries evaluated at z^0 .

THEOREM 1: If $[\partial w / \partial z]^0$ is positive semi-definite, and if z^0 is optimal for (1), then $z^{0T} w(z^0) = 0$.

PROOF: The hypothesis that z^0 is optimal for (1) implies [8, Theorem 1]

that there exists an N-vector λ such that

$$\begin{aligned} w(z^0) + [\partial w / \partial z]^0 (z^0 - \lambda) &\geq 0 \\ z^{0T} \{ w(z^0) + [\partial w / \partial z]^0 (z^0 - \lambda) \} &= 0 \\ \lambda^T w(z^0) &= 0 \end{aligned}$$

Utilizing these facts, the feasibility of z^0 , and the positive semi-definiteness of $[\partial w / \partial z]^0$, we obtain

$$0 \leq z^{0T} w(z^0) = z^{0T} [\partial w / \partial z]^0 (\lambda - z^0) \leq (z^0 - \lambda)^T [\partial w / \partial z]^0 (\lambda - z^0) \leq 0$$

Thus, equality holds throughout and the proof is complete.

We make no claim that these are the weakest conditions* under which the theorem is true; nevertheless, they are easily stated. Under the hypotheses set forth above, Equation (2) becomes an optimality criterion for a feasible vector z^0 in the program (1).

Some of the familiar statements of duality in linear and nonlinear programming, see [3] and [6], can be derived from twice continuously differentiable functions $K(x, y)$ which are strictly convex in $x \geq 0$ and strictly concave in $y \geq 0$. (The strictness can be dropped when the function $K(x, y)$ is of the quadratic type (5) shown below.) Indeed, associated with $K(x, y)$ is a pair of dual programs, see [3].

* For example, for a positive-definite Jacobian matrix, the result can be obtained by applying Fritz John's Theorem 1 [7].

PRIMAL		DUAL	
Minimize	$K - y^T [\partial K / \partial y]$	Maximize	$K - x^T [\partial K / \partial x]$
subject to	$[\partial K / \partial y] \leq 0$	subject to	$[\partial K / \partial x] \geq 0$
	$x \geq 0$		$x \geq 0$
	$y \geq 0$		$y \geq 0$

Let

$$(3) \quad \begin{aligned} u(x,y) &= [\partial K / \partial x] \geq 0, \quad v(x,y) = -[\partial K / \partial y] \geq 0 \\ z &= (x,y), \quad w(z) = (u(x,y), v(x,y)) \end{aligned}$$

We seek a $z^0 \geq 0$ such that $w(z^0) \geq 0$ and $z^{0T} w(z^0) = 0$, for such a z^0 is feasible in both the primal and the dual and makes their objective functions equal. This condition is also necessary as may be derived from the duality theorem [3] and also by considering the program:

$$(4) \quad \begin{aligned} &\text{Minimize} \quad x^T [\partial K / \partial x] - y^T [\partial K / \partial y] \\ &\text{subject to} \quad \begin{cases} [\partial K / \partial x] \geq 0 \\ -[\partial K / \partial y] \geq 0 \\ x \geq 0 \\ y \geq 0 \end{cases} \end{aligned}$$

which clearly has the form (1) under the identifications (3). Theorem 1 applies in the present case since

$$[\partial w / \partial z] = \begin{pmatrix} \partial^2 K / \partial x^2 & -\partial^2 K / \partial y \partial x \\ \partial^2 K / \partial x \partial y & -\partial^2 K / \partial y^2 \end{pmatrix}$$

and for all $(n+m)$ -vectors λ ,

$$\lambda^T [\partial w / \partial z] \lambda = \lambda^T \begin{pmatrix} \partial^2 K / \partial x^2 & -\partial^2 K / \partial y \partial x \\ \partial^2 K / \partial x \partial y & -\partial^2 K / \partial y^2 \end{pmatrix} \lambda = \lambda^T \begin{pmatrix} \partial^2 K / \partial x^2 & 0 \\ 0 & -\partial^2 K / \partial y^2 \end{pmatrix} \lambda \geq 0$$

because $[\partial^2 K / \partial x^2]$ and $-\partial^2 K / \partial y^2$ are positive semi-definite due to the convexity and concavity properties of $K(x, y)$, respectively.

As an example of the above, if $g_1(x)$, $0 \leq i \leq m$, are convex functions of $x \in R^n$, the convex program

$$\begin{aligned} &\text{Minimize} && g_0(x) \\ &\text{subject to} && g_1(x) \leq 0 && (1 \leq i \leq m) \\ &&& x \geq 0 \end{aligned}$$

can be derived from $K(x, y) = g_0(x) + \sum_{i=1}^m y_i g_i(x)$. Another example is provided by a function of the form

$$(5) \quad K(x, y) = \left(\frac{1}{2} x^T D x + c^T x\right) - \left(\frac{1}{2} y^T E y + b^T y\right) - y^T A x,$$

where D and E are symmetric positive semi-definite matrices, c and b are constant vectors, and A is an arbitrary matrix. Symmetric dual quadratic programs result from the formulation suggested above. See [2]. A program of the type (1) can be obtained by forming the combined (self-dual) problem:

$$\begin{aligned} &\text{Minimize} && x^T D x + y^T E y + c^T x + b^T y \\ &\text{subject to} && D x - A^T y + c \geq 0 \\ &(6) && A x + E y + b \geq 0 \\ &&& x \geq 0 \\ &&& y \geq 0 \end{aligned}$$

With $u = D x - A^T y + c$, $v = A x + E y + b$, $w = (u, v)$ and $z = (x, y)$, a solution

z is sought such that $z \geq 0$, $w \geq 0$, and $z^T w = 0$. Such a vector solves the program (6) as well as both of the programs out of which it was formed.

We shall now study properties of complementary basic solutions of the linear system

$$(7) \quad Iw - Mz = q$$

where $w = (w_1, \dots, w_N)^T$ and $z = (z_1, \dots, z_N)^T$ are real variable vectors, $q = (q_1, \dots, q_N)^T$ is known, and M is any $N \times N$ matrix of coefficients which is either positive semi-definite or positive definite, as specified, but not necessarily symmetric. If the pair $(w, z) \geq 0$ satisfies (7) and the equation $z^T w = 0$, then z is optimal for the quadratic program

$$(8) \quad \begin{aligned} &\text{Minimize} \quad z^T Mz + q^T z \\ &\text{subject to} \quad Mz + q \geq 0 \\ &\quad \quad \quad z \geq 0 \end{aligned}$$

Observe that (6) may be cast in this form by setting $q = (c, b)$ and

$$(9) \quad M = \begin{pmatrix} D & -A \\ A & E \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & E \end{pmatrix} + \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}.$$

In the sum on the right, the first summand is positive semi-definite and the second is skew symmetric. Therefore M is positive semi-definite. For the special case of linear programs, $D = 0$ and $E = 0$, thus $z^T Mz \equiv 0$ for all N -vectors z .

II. Canonical Equivalents

DEFINITION 1: Variable pairs (w_i, z_i) will be called complementary; w_i and z_i will be called complements of each other. A pair (w, z) of N -vectors is a

complementary solution of (7) provided

$$(10) \quad z_1 w_1 = 0, \quad z_2 w_2 = 0, \dots, z_N w_N = 0.$$

DEFINITION 2: A basic set of variables consists of any ordered set of N variables w_i and z_j such that their coefficient matrix in (7), called a basis, is nonsingular.

DEFINITION 3: A complementary basic set of variables is one in which exactly one variable of each complementary pair (w_i, z_i) is basic. (For example, w is a complementary basic set. If M is a singular matrix, then z cannot be basic, although it is complementary. On the other hand, $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is positive semi-definite and $z = (z_1, z_2)^T$ is basic.)

DEFINITION 4: A basic solution is the one found by solving for the values of a given set of basic variables when the nonbasic variables are set equal to zero. (For complementary basic sets, basic solutions are complementary solutions.)

DEFINITION 5: The canonical equivalent of (7) with respect to a basic set of variables having basis B is the system obtained by multiplying through (7) on the left by B^{-1} . We observe that if the basic set of variables is complementary and is redesignated \bar{w} and the nonbasic ones denoted by \bar{z} then this left multiplication results in a new canonical form $\bar{w} = \bar{M} \bar{z} + \bar{q}$.

We observe that (7) itself is in canonical form with respect to the complementary basic set w and that M , the negative of the coefficient matrix of the corresponding nonbasic variables, is at least positive semi-definite. William P. Drews conjectured that the same may be true for \bar{M} the analogue of M appearing in each canonical equivalent of (7) with respect to a complementary basic set of variables. This turned out to be true.

Independently P. Wolfe established the same result for M of form (9) and A. Tucker also established it for any M all of whose principal minors are positive.

THEOREM 2: Given any canonical equivalent of (7) with respect to a complementary basic set, let \bar{M} be the negative of the transformed coefficient matrix of the complementary set of nonbasic variables; then M and \bar{M} are either both positive definite or both positive semi-definite, or both have principal minors with both positive or both non-negative, or both none of these.

PROOF: The proof will be omitted here and will be contained in a separate joint paper with A. Tucker and P. Wolfe.

III. Existence of a Complementary Non-Negative Solution: A Constructive Proof

W.S. Dorn [5] has shown that for a positive definite, but not necessarily symmetric, matrix M there always exists a non-negative complementary solution of (7). That is, there exists a $w^0 \geq 0$ and a $z^0 \geq 0$ such that $Iw^0 - Mz^0 = q$ and $z_i^0 w_i^0 = 0$ for all i . One of the authors [2] extends this result to the positive semi-definite case under the assumption that a non-negative solution to (7) exists. Our present objective is to prove these assertions constructively for matrices with all positive principal minors (and hence Dorn's result as a special case).

THEOREM 3: Given any complementary basic set of variables \bar{w} of system (7) expressed in terms of nonbasic variables $\bar{w} = \bar{M}\bar{z} + \bar{q}$, then \bar{M} has a positive or non-negative diagonal according as M has all principal minors positive or non-negative (or, as a special case, M is positive or positive semi-definite).

PROOF: Let \bar{w} be any complementary basic set of variables selected out of (w, z) and let \bar{z} be the complementary nonbasic variables. Rewriting (7) in canonical form relative to \bar{w} yields $\bar{w} = \bar{M} \bar{z} + \bar{q}$; let M have all positive (non-negative) principal minors, then by Theorem 2 the same is true for \bar{M} and hence $\bar{M} = [a_{ij}]$ has all the entries a_{ii} on the diagonal positive (non-negative).

COROLLARY: Given $\bar{w} = \bar{M} \bar{z} + \bar{q}$, each component \bar{w}_s of \bar{w} is a strictly increasing or nondecreasing function of \bar{z}_s according as M has all principal minors positive or non-negative.

A complementary basic solution will be said to be out-of-kilter if one or more of its basic variables is negative. By the Corollary, if $a_{ss} > 0$ and $\bar{w}_s = \bar{q}_s$ is negative, increasing \bar{z}_s from its current value of zero will cause \bar{w}_s to increase linearly toward zero. Suppose $\bar{w}_s = 0$ at $\bar{z}_s = \bar{z}_s^0 > 0$. In this case, \bar{z}_s can replace \bar{w}_s in the basic set, and its value in the new basic solution will be $\bar{z}_s^0 > 0$. If, during the increase of \bar{z}_s , no other basic variable becomes negative, the resulting basic solution is still complementary and is "less out-of-kilter." However, if some other variable, say \bar{w}_r , goes negative, we will say that the increase of \bar{z}_s is blocked by \bar{w}_r . We shall show that the increase of \bar{z}_s will be unblocked by replacing \bar{w}_r in the basic set by its complement \bar{z}_r .

THEOREM 4: If increasing the value of a nonbasic variable \bar{z}_s causes a basic variable \bar{w}_r to decrease, and if \bar{z}_r can replace \bar{w}_r in the basic set, then, after replacement, \bar{z}_r will increase with increasing \bar{z}_s . For M with all positive principle minors (which includes the case of M positive definite), \bar{z}_r can always replace \bar{w}_r .

PROOF: Let $\bar{M} = [a_{ij}]$. We are assuming $a_{rs} < 0$ and $a_{rr} \geq 0$, where $a_{rr} > 0$ if M has all positive minors, (Theorem 3). To obtain a new canonical equivalent after replacing \bar{w}_r by \bar{z}_r , we reduce the coefficient of \bar{z}_r to a unit column by pivoting on the term $-a_{rr}\bar{z}_r$ in the canonical system $\bar{I}\bar{w} - \bar{M}\bar{z} = \bar{q}$; this results in a transformed matrix $\bar{M}^* = [a_{ij}^*]$ in which $a_{rr}^* = 1/a_{rr} > 0$ and $a_{rs}^* = -a_{rs}/a_{rr} > 0$. It is the fact that $a_{rs}^* > 0$ which implies that \bar{z}_r will increase with increasing \bar{z}_s .

DEFINITION 6: A basic solution is said to be degenerate if one or more of its basic variables is zero.

A standard device (in linear programming) which permits one to proceed as if all basic solutions were nondegenerate is to replace the vector q by a matrix $[q, I]$. The components of w and z then become vectors which are considered to be lexico-positive (lexico-negative) if their first nonzero component (providing such exists) is positive (negative). Otherwise they are zero vectors.

THEOREM 5 (Dantzig, Orden and Wolfe): All basic solutions of (7) are nondegenerate in the lexicographic sense.

PROOF: The proof given in [4] is based on the observation that the vector values of any set of basic variables with basis B are given by the rows of $B^{-1}[q, I] = [B^{-1}q, B^{-1}]$ and each row of B^{-1} has at least one nonzero component.

In the discussions which follow, all ordering relations on the values of basic variables should be interpreted in the lexicographic sense.

THEOREM 6: If one nonbasic variable \bar{z}_s is allowed to take on any vector value and if the other nonbasic variables take on any fixed values in their

first component and zero in the rest, then at most one basic variable can be a zero vector.

PROOF: Set the nonbasic variables other than \bar{z}_s at their fixed values, and let \bar{z}_s be the zero vector. Then the values of the basic variables are non-zero by the same argument as in Theorem 5. If some basic variable \bar{w}_r is to vanish for some other vector value of \bar{z}_s , say $\bar{z}_s = \bar{z}_s^0$, then it must be possible to interchange \bar{w}_r and \bar{z}_s as basic variables. But the new basic solution (which is the same as the prior one obtained by setting $\bar{z}_s = \bar{z}_s^0$) is nondegenerate, so that no other basic variable can have a zero-vector value.

We are now prepared to deal with

THEOREM 7: If M has all principal minors positive, there exists a non-negative complementary solution to (7), i.e., to $Iw - Mz = q$.

Our proof will be constructive. First we introduce some notation:

- i) let $(\bar{w}^0, \bar{z}^0) = (\bar{q}, 0)$ be any complementary basic solution to (7);
- ii) let (\bar{w}^t, \bar{z}^t) be the solution to (7) such that $\bar{z}^t = (0, \dots, 0, \bar{z}_s^t, 0, \dots, 0)$

ITERATIVE PROCEDURE:

Step 0 Set $(\bar{w}, \bar{z}) = (\bar{w}^0, \bar{z}^0)$ and $(w, z) = (q, 0)$ as the starting solution.

Step 1a If $(\bar{w}^0, \bar{z}^0) \geq 0$, terminate; the solution satisfies the theorem.

Step 1b If not, let \bar{z}_s be any \bar{z}_1 such that $\bar{w}_1^0 \leq 0$. Let $t = 0$.

Step 2 Set $\bar{z}_s^{t+1} = \text{Max } \bar{z}_s \geq \bar{z}_s^t$ such that $\bar{w}_s^{t+1} \leq 0$ and $\bar{w}_1^{t+1} \geq 0$ if $\bar{w}_1^t \geq 0$.

Step 3 If $\bar{w}_r^{t+1} = 0$, (i.e., the lexicographic vector \bar{w}_r vanishes) replace \bar{w}_r by \bar{z}_r in the basic set.

Step 4a If $r = s$, return to Step 1a.

Step 4b If $r \neq s$, return to Step 2 with $t + 1$ replacing t .

PROOF: In the interval $\bar{z}_s^t \leq \bar{z}_s \leq \bar{z}_s^{t+1}$, no variable can change sign and become negative (by the definition of \bar{z}_s^{t+1} in Step 2.) Hence v , the number of basic variables with negative values, stays the same or decreases if a negative variable becomes non-negative. The latter is always the case if $r = s$ in Step 4a, because \bar{w}_s^t (originally negative) has been increased to zero value. Therefore there can be only a finite number of returns to Step 1a before all the variables are non-negative. Consider now returns to Step 2. Initially there is a (lexico-) positive increase of \bar{z}_s in an interval $0 \leq \bar{z}_s \leq \bar{z}_s^1$ because of the nondegeneracy of the basic solution. If $r \neq s$, and $\bar{w}_r^t > 0$ for \bar{z}_s^t and $\bar{w}_r^{t+1} = 0$, then $a_{rs} < 0$ and the replacement of \bar{w}_r by \bar{z}_r permits \bar{z}_s to be increased by Theorem 4 and Theorem 6. There can be only a finite number of returns to Step 2 with the same basic set until the largest \bar{z}_s is attained for which $\bar{w}_s < 0$ and v is minimum. Since the number of different basic sets is finite, the proof is complete.

The Case for M Positive Semi-Definite

THEOREM 8: If increasing the value of a nonbasic variable \bar{z}_s causes a basic variable \bar{w}_r to decrease, and if \bar{z}_r cannot replace \bar{w}_r in the basic set, then (\bar{z}_s, \bar{z}_r) can always replace (\bar{w}_s, \bar{w}_r) as basic variables; moreover, increasing \bar{w}_s will increase \bar{z}_r .

PROOF: If \bar{M} is positive semi-definite, it may happen that $a_{rr} = 0$ and consequently that it is impossible to replace \bar{w}_r by \bar{z}_r . In this case, the 2×2 submatrix

$$(11) \quad \bar{M}_2 = \begin{bmatrix} a_{ss} & a_{sr} \\ a_{rs} & a_{rr} \end{bmatrix}$$

has entries satisfying the conditions

$$(12) \quad a_{ss} \geq 0, \quad a_{rr} = 0, \quad a_{sr} + a_{rs} = 0, \quad \text{and} \quad a_{sr} > 0.$$

The third of these, $a_{sr} + a_{rs} = 0$, follows from the fact that $\lambda^T M \lambda = a_{ss} \lambda_1^2 + (a_{sr} + a_{rs}) \lambda_1 \lambda_2$ must be positive semi-definite (which in turn is true because M is positive semi-definite). The positivity of a_{sr} is a consequence of the assumption that \bar{w}_r decreases as \bar{z}_s increases. Since the submatrix \bar{M}_2 is nonsingular, the replacement is possible [9]. Finally, \bar{z}_r (now in the basis) increases with increasing \bar{w}_s (now outside the basis) because $a_{sr} > 0$.

THEOREM 9: No non-negative solution exists if, for some s , all $a_{is} \geq 0$, $a_{ss} = 0$, and $\bar{q}_s < 0$.

PROOF: As in the proof of Theorem 8, if $a_{ss} = 0$ then $a_{s1} = -a_{is} \leq 0$. Therefore equation s consists of all nonpositive coefficients and a negative constant term, and hence is not solvable in non-negative variables.

In developing an analogue of the positive definite case, we note that if the conditions of Theorem 9 hold, \bar{z}_s can be increased indefinitely without making any \bar{w}_1 change sign and become negative. In the example (13) below, $q_1 < 0$ and z_1 can be increased as much as desired without inducing any sign changes in the basic variables. Letting $z_1 \rightarrow +\infty$ we have $w_1 = -1$, $w_2 = 1$, $w_3 = 1 + z_1 \rightarrow +\infty$, and $w_4 = -1 - z_1 \rightarrow -\infty$. This problem is, however, solvable.

(13)

w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	q
1	0	0	0	0	0	1	-1	-1
0	1	0	0	0	0	1	1	1
0	0	1	0	-1	-1	0	0	1
0	0	0	1	1	-1	0	0	-1

The device we have used to prevent the values of basic variables from tending to $-\infty$ is to impose a uniform negative lower limit ℓ for variables with negative values. (A nonuniform lower limit will do as well.) Once a variable rises into the non-negative range, its admissible lower limit is changed back to zero. If we start with the complementary basic solution $(w, z) = (q, 0)$, we may set $\ell = \min q_i$. (If $\min q_i \geq 0$, then $(q, 0)$ solves the problem.) Since $z_i = 0$ initially, $z_i \geq 0$ will hold in all subsequent iterations. From these considerations, it follows that at most one variable of each complementary pair can ever be negative. If a basic variable \bar{w}_r (with negative value) decreases to ℓ when \bar{z}_s is increased, \bar{w}_r is to be replaced in the basic set; it becomes a nonbasic variable with value $\bar{w}_r = \ell$.

We shall modify the notations of Theorem 7 to treat the positive semi-definite case.

(i') Define (\bar{w}^0, \bar{z}^0) to be any solution of (7) such that $\bar{z}_i^0 = 0$ or ℓ , where $\bar{z}_i^0 = \ell$ implies $\bar{w}_i^0 \geq 0$.

(ii') Define (\bar{w}^t, \bar{z}^t) to be the solution to (7) such that $\bar{z}^t = \bar{z}^0 + (0, \dots, 0, \bar{z}_s^t - \bar{z}_s^0, 0, \dots, 0)$ for some \bar{z}_s^t .

MODIFIED ITERATIVE PROCEDURE:

Step 0' Set $(\bar{w}, \bar{z}) = (\bar{w}^0, \bar{z}^0)$ and $(w, z) = (q, 0)$ as the starting solution.

Step 1a' If $(\bar{w}^0, \bar{z}^0) \geq 0$, terminate. The solution satisfies the theorem.

Step 1b' Let \bar{z}_s be any \bar{z}_i such that $\bar{w}_i^0 < 0$ or $\bar{z}_i^0 = \ell$; let $t = 0$.

Step 2a' Set $\bar{z}_s^{t+1} = \max \bar{z}_s \geq \bar{z}_s^t$ such that $((\bar{w}_s^{t+1} \leq 0 \text{ if } \bar{w}_s^t < 0 \text{ or } \bar{z}_s^{t+1} \leq 0 \text{ if } \bar{z}_s^t < 0) \text{ and } (\bar{w}_i^{t+1} \geq 0 \text{ if } \bar{w}_i^t \geq 0))$.

Step 2b' If $\bar{z}_s^{t+1} = +\infty$, terminate. No feasible solution exists.

Step 2c' If $\bar{z}_s^{t+1} = 0$, return to Step 1a'.

Step 3a' If $\bar{w}_r = 0$ or ℓ , and $a_{rr} \neq 0$, replace \bar{w}_r by \bar{z}_r in the basic set.

Step 3b' If $\bar{w}_r = 0$ or ℓ , and $a_{rr} = 0$, replace (\bar{w}_s, \bar{w}_r) by (\bar{z}_s, \bar{z}_r) in the basic set.

Step 4a' If $r = s$, return to Step 1a'.

Step 4b' If $r \neq s$, return to Step 2a' with $t + 1$ replacing t .

THEOREM 10: If \bar{z}_r in the basic set increases with increasing \bar{z}_s outside the basic set, then the same class of solutions is generated by putting \bar{z}_s in the basic set in place of \bar{z}_r and increasing \bar{z}_r .

PROOF: There is only one degree of freedom and the relations are linear.

Paraphrase Summary of the Algorithm

We may thus simplify the foregoing procedure into saying in effect:

- (0) Increase \bar{z}_s until some \bar{w}_{r_1} drops to 0 (or ℓ), replace \bar{w}_{r_1} by \bar{z}_s ,
- (1) increase \bar{z}_{r_1} until some \bar{w}_{r_2} drops to 0 (or ℓ), replace \bar{w}_{r_2} by \bar{z}_{r_1} ,
- .
- .
- (k) increase $\bar{z}_{r_{k-1}}$ until some \bar{w}_{r_k} drops to 0 (or ℓ), replace \bar{w}_{r_k} by $\bar{z}_{r_{k-1}}$;
- (k+1) increase \bar{z}_{r_k} until \bar{w}_s increases to 0, replace \bar{w}_s by \bar{z}_{r_k} , where we denote by \bar{w}_1 any variable in the current basic set — although it may be the same as some \bar{z}_1 which was earlier not in the basic set.

From these discussions, the following observations can be made:

THEOREM 11 (Cottle [2]): If $Iw - Mz = q$ is solvable in non-negative variables where M is positive semi-definite, there exists a non-negative complementary solution.

NOTE: The positive definite case was shown by Dorn [5].

THEOREM 12: $Iw - Mz = q$ is not solvable in non-negative variables if and only if the representation of some column s in terms of some complementary basis has nonpositive weights with zero weight on the complementary column whereas the representation for q has a negative weight for this column.

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